

# ON THE DENSITY OF THE ODD VALUES OF THE PARTITION FUNCTION

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**ABSTRACT.** The purpose of this note is to introduce a new approach to the study of one of the most basic and seemingly intractable problems in partition theory, namely the conjecture that the partition function  $p(n)$  is equidistributed modulo 2.

Our main result will relate the densities, say  $\delta_t$ , of the odd values of the  $t$ -multipartition functions  $p_t(n)$ , for several integers  $t$ . In particular, we will show that if  $\delta_t > 0$  for some  $t \in \{5, 7, 11, 13, 17, 19, 23, 25\}$ , then  $\delta_1 > 0$ ; that is,  $p(n)$  itself is odd with positive density. Notice that, currently, the best unconditional result does not even imply that  $p(n)$  is odd for  $\sqrt{x}$  values of  $n \leq x$ . In general, we conjecture that  $\delta_t = 1/2$  for all  $t$  odd, i.e., that similarly to the case of  $p(n)$ , all multipartition functions are in fact equidistributed modulo 2.

Our arguments will employ a number of algebraic and analytic methods, ranging from an investigation modulo 2 of some classical Ramanujan identities and several other eta product results, to a unified approach that studies the parity of the Fourier coefficients of a broad class of modular form identities recently introduced by Radu.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $p(n)$  denote the number of *partitions* of a nonnegative integer  $n$ , i.e., the number of ways  $n$  can be written as  $n = \lambda_1 + \cdots + \lambda_k$ , for integers  $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ . It is well known (see e.g. [3]) that the generating function for  $p(n)$  is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{i=1}^{\infty} (1 - q^i)}.$$

Over the years, an enormous amount of research, which has developed or employed a variety of combinatorial, algebraic or analytic techniques, has been devoted to the problem of studying the behavior of the partition function  $p(n)$  modulo a prime number  $p$  (see for instance, as a highly nonexhaustive list, [2, 19, 22, 27, 29, 31]). While a significant body of literature is now available on the properties of  $p(n)$  modulo  $p$  for  $p \geq 5$  (i.e., when  $p$  does not divide 24), precious little is known today on the behavior of  $p(n)$  modulo 2 or 3. The present work focuses on the case  $p = 2$ .

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Even though infinitely many values of  $p(n)$  must be odd, we are still very far from knowing whether the partition function is odd with *positive density*; i.e., if we define

$$\delta_1 = \lim_{x \rightarrow \infty} \frac{\#\{n \leq x : p(n) \text{ is odd}\}}{x},$$

if the limit exists, then proving that  $\delta_1 > 0$  seems to be well beyond the current technology. In fact, the best asymptotic lower bound known today on the number of odd values of  $p(n)$  for  $n \leq x$ , due to Bellaïche and Nicolas [4] (who have improved results of several other authors; see, just as a sample, [1, 11, 21, 24, 32]), is given by  $\frac{\sqrt{x}}{(\log x)^{7/8}}$ , for  $x \rightarrow \infty$ . (Tangentially, we know again from [4] that the number of  $n \leq x$  such that  $p(n)$  is even is at least of order  $\sqrt{x} \log \log x$ ; notice also that, unlike for the odd values, an asymptotic lower bound of order  $\sqrt{x}$  for the even values is very easy to obtain.)

It is conjectured that  $\delta_1$  exists and equals  $1/2$  (see e.g. [7, 26]), and indeed, it is widely believed that  $p(n)$  behaves essentially “randomly” modulo 2.

For any positive integer  $t$ , define  $p_t(n)$  as the number of  $t$ -multipartitions of  $n$ ; i.e., the integers  $p_t(n)$  are the coefficients of the  $t$ -th power of the generating function of  $p(n)$ :

$$\sum_{n=0}^{\infty} p_t(n) q^n = \frac{1}{\prod_{i=1}^{\infty} (1 - q^i)^t}.$$

The ordinary partition function is therefore  $p(n) = p_1(n)$ . Likewise, denote by  $\delta_t$  the density of the odd values of  $p_t(n)$ :

$$\delta_t = \lim_{x \rightarrow \infty} \frac{\#\{n \leq x : p_t(n) \text{ is odd}\}}{x},$$

if this limit exists. Similarly to the case of  $p(n)$ , showing that  $\delta_t$  exists and is positive also seems an intractable problem, for any value of  $t$ . Currently, the best asymptotic lower bound on the number of odd values of  $p_t(n)$  with  $n \leq x$  is given by  $\frac{\sqrt{x}(\log \log x)^k}{\log x}$ , for any fixed  $k$  (see the third author [37], also for a slightly weaker elementary bound when  $t = 3$ ); hence, notice that a bound of  $\sqrt{x}$  is not known yet for any  $t \geq 1$ .

In general, extending the conjecture that  $\delta_1 = 1/2$ , we believe that  $\delta_t = 1/2$  for all odd positive values of  $t$ ; in other words, that *all  $t$ -multipartition functions are equidistributed modulo 2*. Our conjecture is supported by substantial computer evidence. We have:

**Conjecture 1.**  $\delta_t$  exists and equals  $\frac{1}{2}$ , for any odd positive integer  $t$ . Equivalently, if  $t = 2^k t_0$  with  $t_0 \geq 1$  odd, then  $\delta_t$  exists and equals  $2^{-k-1}$ .

(Notice that, because of obvious parity reasons, it suffices to assume that  $t$  be odd.) Our main goal is to introduce a new approach to the study of the parity of  $p(n)$ , by connecting it to the parity of  $p_t(n)$ , for several values of  $t$ . In order to do this, we will employ a number of algebraic and analytic methods described below. The next theorem is the main result of this

paper. It will follow as a corollary of certain partition congruences of independent interest, which we will state in the subsequent theorems.

**Theorem 2.**  $\delta_t > 0$  implies  $\delta_1 > 0$  for

$$t = 5, 7, 11, 13, 17, 19, 23, 25.$$

Moreover,  $\delta_t > 0$  implies  $\delta_r > 0$  for the following pairs  $(t, r)$ :

$$(27, 9), (9, 3), (25, 5), (15, 3), (21, 3), (27, 3).$$

(Notice that the case  $(27, 3)$  follows by combining  $(27, 9)$  and  $(9, 3)$ , and  $(25, 1)$  follows from  $(25, 5)$  and  $(5, 1)$ .) In particular, Theorem 2 says that if the coefficients of the  $t$ -th power of the generating function of  $p(n)$  are odd with positive density for some  $t \in \{5, 7, 11, 13, 17, 19, 23, 25\}$ , then  $p(n)$  itself is odd with positive density — a fact that, as we saw earlier, seems virtually impossible to show unconditionally today.

In general, the relationship between the density of the odd coefficients of a series in  $\mathbb{F}_2[[q]]$  and the density of the odd coefficients of one of its (odd) powers is not obvious. A simple example is given by  $F(q) = \prod_{i=0}^{\infty} (1 + q^{4^i})$ , whose nonzero coefficients are supported on the numbers with binary digits all in even places (and therefore have density zero), while it is easy to see that, modulo 2,  $F(q)^3 \equiv \sum_{n=0}^{\infty} q^n$  (of course giving density 1).

We will be able to prove many of the results of Theorem 2 by algebraic means, by studying a number of congruences modulo 2, including a simple application of the so-called Ramanujan’s “most beautiful identity,” i.e., a generating function identity for  $p(5n + 4)$ . This will be done in the next section. We will also show a few other related results of independent interest in that section. For instance, applying the same methods, we will provide a short proof, and in a sense an explanation, of the “striking result” of a recent paper of Hirschhorn and Sellers [15], which in turn improved the seven-author paper [8] — namely that the odd coefficients of 5-regular partitions have density at most  $1/4$ . (In fact, we will relate the parity of 5-regular partitions with that of 20-regular partitions.) Finally, in the last section of the paper, we will show all of the congruences involved in our main theorem, including those that we have not been able to prove algebraically, by a unified approach to the parity of the Fourier coefficients of a broad class of modular forms recently introduced by Radu [28].

Theorem 2 is a consequence of the next two results, which we will spend the bulk of our paper proving. Recall that two series  $f(q) = \sum_{n=n_1}^{\infty} a(n)q^n$  and  $g(q) = \sum_{n=n_2}^{\infty} b(n)q^n$  are said to satisfy  $f(q) \equiv g(q) \pmod{m}$ , if  $a(n) \equiv b(n) \pmod{m}$  for all integers  $n$ . All congruences in this work will be modulo 2, unless otherwise stated.

**Theorem 3.** *The congruence*

$$q \sum_{n=0}^{\infty} p_t(an + b)q^n \equiv \frac{1}{\prod_{i=1}^{\infty} (1 - q^i)^{at}} + \frac{1}{\prod_{i=1}^{\infty} (1 - q^{ai})^t}$$

*holds for the following twelve triples  $(a, b, t)$ :*

$$(5, 4, 1), (7, 5, 1), (11, 6, 1), (13, 6, 1), (17, 5, 1), (19, 4, 1), \\ (23, 1, 1), (3, 2, 3), (5, 2, 3), (7, 1, 3), (5, 0, 5), (3, 0, 9).$$

**Theorem 4.** *The congruence*

$$q^2 \sum_{n=0}^{\infty} p_t(a^2n + b)q^n \equiv \frac{1}{\prod_{i=1}^{\infty} (1 - q^i)^{a^2t}} + \frac{1}{\prod_{i=1}^{\infty} (1 - q^{ai})^{at}} + \frac{q}{\prod_{i=1}^{\infty} (1 - q^i)^t}$$

*holds for the following two triples  $(a, b, t)$ :*

$$(3, 8, 3), (5, 24, 1).$$

The proof of Theorem 2 is easily deduced once Theorems 3 and 4 are established, as we explain now for  $\delta_5 > 0$  implying  $\delta_1 > 0$ ; the logic in the other cases is similar.

*Proof.* Suppose we know Theorem 3 for case  $(5, 4, 1)$ ; i.e.,

$$(1) \quad q \sum_{n=0}^{\infty} p(5n + 4)q^n \equiv \frac{1}{\prod_{i=1}^{\infty} (1 - q^i)^5} + \frac{1}{\prod_{i=1}^{\infty} (1 - q^{5i})}.$$

Assuming that  $\delta_5 > 0$ , then the densities of the odd coefficients of  $\sum_{n=0}^{\infty} p(5n + 4)q^n$  and  $1/\prod_{i=1}^{\infty} (1 - q^{5i}) = \sum_{n=0}^{\infty} p(n)q^{5n}$  cannot both be zero. This immediately suffices to conclude that  $\delta_1 > 0$ , since if the density is positive for  $p(5n + 4)$ , then it clearly is for  $p(n)$  as well.  $\square$

Notice that a number of additional results can also follow from these congruences. Some are likely counterfactual but might serve as hypothesis testing or sources of absurdities in future investigations. For instance, assuming  $\delta_1 = 1$ , one would obtain the following.

**Proposition 5.** *If  $\delta_1 = 1$ , then  $\delta_5 = 4/5$ , with density zero for the odd coefficients of the series  $\sum_{n=0}^{\infty} p_5(5n)q^{5n}$  and density 1 among all other coefficients.*

*Proof.* If  $\delta_1 = 1$ , then in (1), the left-hand-side will be odd with density 1, since  $5n + 4$  is a subprogression of  $n$ . On the right-hand-side, the coefficients of  $q^{5n}$  in the second term will be odd with density 1, and therefore the same coefficients in the first term must be odd with density zero. Outside of this progression, however, the coefficients of the first term must be odd with density 1, immediately giving the desired conclusion.  $\square$

Notice that, as it seems highly likely to be the case, it follows that *if the odd density of  $p_5(n)$  does not satisfy the conclusions of Proposition 5, then  $\delta_1 < 1$ , and thus the even coefficients of  $p(n)$  have positive density* — a fact that, as we mentioned earlier, is widely believed to be true but appears to be well beyond the current state of the art.

The converse of Theorem 2, i.e., for instance showing that  $\delta_1 > 0$  implies  $\delta_5 > 0$ , is not apparent and seems to require more knowledge of the structure of any odd density of  $p(n)$ . One can obtain, just as an example, the following weaker result:

**Proposition 6.**  $\delta_5 = 0$  implies  $\delta_1 \leq \frac{5}{6}$ , with almost all of the odd values of  $p(5n + 4)$  found in the  $p(25n + 24)$  subprogression.

*Proof.* Let  $x$  denote the odd density of  $p(5n + 4)$ , and  $y$  the odd density of the partition function outside the  $5n + 4$  progression. Clearly,  $\delta_1 = 4y/5 + x/5$ . Assume  $\delta_5 = 0$ . Thus, by (1), it easily follows that  $x = \delta_1/5$ , and therefore  $\delta_1 = 5y/6 \leq 5/6$ .

Moreover, we have from (1) that  $p(25k + 24)$  must contain almost all of the odd values of  $p(5n + 4)$ , since the second term on the right-hand-side contributes even values to all other subprogressions of  $5n + 4$ , while by the assumption that  $\delta_5 = 0$ , the first term is odd with density zero. This completes the proof.  $\square$

## 2. ALGEBRAIC PROOFS

In this section, we show algebraically the cases  $(5, 4, 1)$ ,  $(7, 5, 1)$ ,  $(13, 6, 1)$ ,  $(3, 2, 3)$ ,  $(5, 2, 3)$ ,  $(5, 0, 5)$ , and  $(3, 0, 9)$  of Theorem 3, and both cases  $(3, 8, 3)$  and  $(5, 24, 1)$  of Theorem 4. We have not been able to determine algebraic proofs for the cases  $(11, 6, 1)$ ,  $(17, 5, 1)$ ,  $(19, 4, 1)$ ,  $(23, 1, 1)$ , and  $(7, 1, 3)$  of Theorem 3, so these will only be shown analytically in the next section.

We employ the standard  $q$ -series notation

$$\prod_{i=0}^{\infty} (1 - aq^i) = (a; q)_{\infty}; \quad (q; q)_{\infty} = (q)_{\infty}.$$

First we prove Theorem 3 for  $(5, 4, 1)$ , which is perhaps the easiest case.

*Proof.* Ramanujan's "most beautiful identity" states that

$$(2) \quad \sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \frac{(q^5)_{\infty}^5}{(q)_{\infty}^6}.$$

Now, using a congruence of Blecksmith-Brillhart-Gerst (see [6], p. 301; cf. also Hirschhorn's equation (13) in [13]), we can easily obtain

$$(3) \quad \frac{(q)_{\infty}}{\prod_{i=1}^{\infty} (1 - q^{10i-5})} \equiv (q)_{\infty} (q^5)_{\infty} \equiv \sum_{n=1}^{\infty} q^{n^2-n} + \sum_{n=1}^{\infty} q^{5n^2-5n+1}.$$

It is well known that

$$(q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} \equiv \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and therefore we may transform equation (3) into

$$(4) \quad (q)_\infty (q^5)_\infty \equiv (q)_\infty^6 + q (q^5)_\infty^6.$$

It now follows by (2) and standard algebraic manipulations that

$$(5) \quad q \sum_{n=0}^{\infty} p(5n+4) q^n \equiv q \frac{(q^5)_\infty^5}{(q)_\infty^6} \equiv \frac{1}{(q)_\infty^5} + \frac{1}{(q^5)_\infty},$$

which is the desired result.  $\square$

As a further and simple application of congruence (4), we next give a quick proof of the main result of Hirschhorn-Sellers [15], which in turn improved the work of [8] on 5-regular partitions. Recall that a partition of  $n$  is  $m$ -regular if it does not contain parts that are multiples of  $m$ . One usually denotes the number of  $m$ -regular partitions of  $n$  by  $b_m(n)$ , and it easily follows from the definition that the generating function for  $b_m(n)$  is given by:

$$\sum_{n=0}^{\infty} b_m(n) q^n = \frac{(q^m)_\infty}{(q)_\infty}.$$

Denote by  $\delta^{[m]}$  the density of the odd values of  $b_m$ . While it is known that, for certain  $m$ ,  $\delta^{[m]} = 0$  (for instance, when  $m = 2^a \cdot m_0$  with  $m_0 \leq \sqrt{m}$ ; see Ono-Gordon [25] and also Serre [30]), for other values of  $m$  it seems reasonable to believe that  $\delta^{[m]} > 0$ , though there exists no  $m$  yet for which this has been established. (In fact, we simply remark here that if  $\delta^{[m]} > 0$  for some  $m$ , then it is easy to see that, asymptotically,  $p(n)$  is odd for at least the order of  $\sqrt{x}$  values of  $n \leq x$  — a fact that, as we saw, is also not known.)

Currently, the best bound for  $\delta^{[5]}$  is the “striking result”  $\delta^{[5]} \leq 1/4$ , obtained in [15]. The next theorem consists of a quick proof, and in a sense an explanation, of this fact, in a way that also nicely relates it to the density of 20-regular partitions.

**Theorem 7.** *We have  $\delta^{[5]} = \delta^{[20]}/4$ . In particular,  $\delta^{[5]} \leq 1/4$ .*

*Proof.* By multiplying both sides of (4) by  $1/(q)_\infty^2$ , we obtain

$$(6) \quad \sum_{n=0}^{\infty} b_5(n) q^n \equiv (q)_\infty^4 + q \frac{(q^5)_\infty^6}{(q)_\infty^2}.$$

Similarly, multiplication by  $(q^5)_\infty^2/(q)_\infty^2$  across (4) yields

$$(7) \quad \frac{(q^5)_\infty^3}{(q)_\infty} \equiv (q)_\infty^4 (q^5)_\infty^2 + q \frac{(q^5)_\infty^8}{(q)_\infty^2}.$$

Thus, combining (6) and (7), we get

$$(8) \quad \sum_{n=0}^{\infty} b_5(n)q^n \equiv (q)_{\infty}^4 + q(q)_{\infty}^8(q^5)_{\infty}^4 + q^3 \frac{(q^5)_{\infty}^{16}}{(q)_{\infty}^4} \equiv (q)_{\infty}^4 + q(q)_{\infty}^8(q^5)_{\infty}^4 + \sum_{n=0}^{\infty} b_{20}(n)q^{4n+3}.$$

Hence, one moment's thought now gives that the theorem follows if we show that the odd coefficients of  $(q)_{\infty}^4 + q(q)_{\infty}^8(q^5)_{\infty}^4$  have density zero. This is clearly the case for the first summand, since by the Pentagonal Number Theorem,

$$(q)_{\infty}^4 \equiv \sum_{n \in \mathbb{Z}} q^{2n(3n-1)}.$$

As for  $q(q)_{\infty}^8(q^5)_{\infty}^4$ , again by the Pentagonal Number Theorem we have

$$q(q)_{\infty}^8(q^5)_{\infty}^4 \equiv q \sum_{m \in \mathbb{Z}} q^{4m(3m-1)} \sum_{n \in \mathbb{Z}} q^{10n(3n-1)}.$$

But it is a classical result of Landau (see [17] or for instance Serre [31]) that the integers that can be represented by a quadratic form in two variables have density zero. Since this obviously remains true once we reduce modulo 2, the proof of the theorem is complete.  $\square$

**Remark 8.** (1) Computer calculations suggest that it seems reasonable to conjecture that  $\delta^{[5]} = 1/8$  (or equivalently, that  $\delta^{[20]} = 1/2$ ).

(2) We remark here without proof that similar nice applications to  $m$ -regular partitions can also be given for other values of  $m$ . For instance, from identity (10) below, it is possible to deduce in an analogous fashion that  $\delta^{[7]} = \delta^{[28]}/2$  (and, consequently, that  $\delta^{[7]} \leq 1/2$ ).

We now show the remaining algebraic cases.

*Proof.* We begin with Theorem 3 for  $(13, 6, 1)$ . Zuckerman's classical identity for  $p(13n + 6)$  [38] easily implies that

$$(9) \quad \sum_{n=0}^{\infty} p(13n + 6)q^n \equiv \frac{(q^{13})_{\infty}}{(q)_{\infty}^2} + q^5 \frac{(q^{13})_{\infty}^{11}}{(q)_{\infty}^{12}} + q^6 \frac{(q^{13})_{\infty}^{13}}{(q)_{\infty}^{14}},$$

while from Calkin *et al.* [8], we can see that

$$\frac{(q^{13})_{\infty}}{(q)_{\infty}} + (q)_{\infty}^{12} \equiv q(q)_{\infty}^{10}(q^{13})_{\infty}^2 + q^6(q^{13})_{\infty}^{12} + q^7 \frac{(q^{13})_{\infty}^{14}}{(q)_{\infty}^2}.$$

Thus, now multiply both sides of the latter congruence by

$$\frac{1}{(q)_{\infty}^{12}(q^{13})_{\infty}},$$

and then substitute into (9). This yields the desired result.

Case (7, 5, 1) can be proved using the next identity, which is essentially due to Ramanujan (see [5], equation (24.6a)):

$$\sum_{n=0}^{\infty} p(7n+5)q^n \equiv \frac{(q^7)_{\infty}^3}{(q)_{\infty}^4} + q \frac{(q^7)_{\infty}^7}{(q)_{\infty}^8},$$

along with the following, which is equivalent to an identity of Lin (cf. [18], equation 2.4):

$$(10) \quad (q)_{\infty}(q^7)_{\infty} \equiv (q)_{\infty}^8 + q(q)_{\infty}^4(q^7)_{\infty}^4 + q^2(q^7)_{\infty}^8.$$

Divide now through identity (10) by  $(q)_{\infty}^8(q^7)_{\infty}$ , and substitute into the previous one to complete the proof.

Case (3, 2, 3) can be shown in a similar fashion, starting from an identity of Chan ([9], Theorem 1; see also Xiong [34], Theorem 1.1), which has the immediate corollary

$$(11) \quad \sum_{n=0}^{\infty} p_3(3n+2)q^n \equiv \frac{(q^3)_{\infty}^9}{(q)_{\infty}^{12}}.$$

We can combine this with [14], Theorem 2.1 (note that a power of 2 is missing from the factor  $(q^{12})_{\infty}$  in the published version of this paper), which yields

$$(12) \quad \frac{1}{(q)_{\infty}^9(q^3)_{\infty}^9} \equiv \frac{q}{(q)_{\infty}^{12}} + \frac{1}{(q^3)_{\infty}^{12}}.$$

Indeed, if we multiply across this latter by  $(q^3)_{\infty}^9$ , we easily obtain the desired result.

Case (5, 2, 3) of Theorem 3 is the claim that

$$q \sum_{n=0}^{\infty} p_3(5n+2)q^n \equiv \frac{1}{(q)_{\infty}^{15}} + \frac{1}{(q^5)_{\infty}^3}.$$

We begin with an identity of Chan-Lewis ([10], identity (1.11); see also Xiong [35]), which modulo 2 becomes

$$q \sum_{n=0}^{\infty} p_3(5n+2)q^n \equiv q \frac{(q^5)_{\infty}^3}{(q)_{\infty}^6} + q^2 \frac{(q^5)_{\infty}^9}{(q)_{\infty}^{12}} + q^3 \frac{(q^5)_{\infty}^{15}}{(q)_{\infty}^{18}}.$$

The third term on the right-hand-side is the cube of  $q \sum_{n=0}^{\infty} p(5n+4)q^n$ , which by the (5, 4, 1) case of Theorem 3, is congruent to  $\frac{1}{(q)_{\infty}^5} + \frac{1}{(q^5)_{\infty}}$ . The middle term is  $\frac{1}{(q^5)_{\infty}}$  times the square of  $q \sum_{n=0}^{\infty} p(5n+4)q^n$ , and the first term is  $\frac{1}{(q^5)_{\infty}^2}$  times  $q \sum_{n=0}^{\infty} p(5n+4)q^n$  itself, again using Theorem 3 for (5, 4, 1).

Therefore, now expand the necessary powers of  $\frac{1}{(q)_{\infty}^5} + \frac{1}{(q^5)_{\infty}}$  and cancel the even terms; the remaining terms easily constitute the desired identity.

Case (5, 0, 5) of Theorem 3 requires the (5, 24, 1) case of Theorem 4, so we prove the latter first. This is the claim that

$$q^2 \sum_{n=0}^{\infty} p(25n+24)q^n \equiv \frac{1}{(q)_{\infty}^{25}} + \frac{1}{(q^5)_{\infty}^5} + \frac{q}{(q)_{\infty}}.$$



We will show it using Zuckerman's identity [38] for  $\sum_{n=0}^{\infty} p(25n+24)q^n$  (or alternatively, [5], equation (21.1)), which, modulo 2, gives us that

$$(13) \quad \sum_{n=0}^{\infty} p(25n+24)q^n \equiv \frac{(q^5)_{\infty}^6}{(q)_{\infty}^7} + q^2 \frac{(q^5)_{\infty}^{18}}{(q)_{\infty}^{19}} + q^4 \frac{(q^5)_{\infty}^{30}}{(q)_{\infty}^{31}},$$

along with a repeated application of our recently proven fact (see the case  $(5, 4, 1)$ ):

$$(14) \quad q \frac{(q^5)_{\infty}^5}{(q)_{\infty}^6} \equiv \frac{1}{(q)_{\infty}^5} + \frac{1}{(q^5)_{\infty}}.$$

Indeed, let us multiply through (13) by  $q^2$ , then treat the third term on the right-hand-side as  $(q)_{\infty}^5$  times the 6th power of (14), the second term as  $(q)_{\infty}^5 / (q^5)_{\infty}^2$  times the 4th power of (14), and the first term as  $q(q^5)_{\infty} / (q)_{\infty}$  times (14) itself. Finally, cancel what terms are possible to eventually obtain:

$$q^2 \sum_{n=0}^{\infty} p(25n+24)q^n \equiv \frac{q}{(q)_{\infty}} + \frac{1}{(q)_{\infty}^{25}} + q \frac{(q^5)_{\infty}}{(q)_{\infty}^6} + \frac{1}{(q)_{\infty}^5 (q^5)_{\infty}^4}.$$

If now we again apply (14) (dividing by  $(q^5)_{\infty}^4$ ) to the third term on the right-hand-side of (13) and then cancel, we finally obtain the desired congruence.

The  $(5, 0, 5)$  case of Theorem 3 is the claim that

$$q \sum_{n=0}^{\infty} p_5(5n)q^n \equiv \frac{1}{(q)_{\infty}^{25}} + \frac{1}{(q^5)_{\infty}^5}.$$

In order to prove this, use the  $(5, 24, 1)$  identity and divide through by  $q$ , to transform the claim into

$$(15) \quad \sum_{n=0}^{\infty} p_5(5n)q^n \equiv q \sum_{n=0}^{\infty} p(25n+24)q^n + \frac{1}{(q)_{\infty}}.$$

Recall from the case  $(5, 4, 1)$  that

$$q \sum_{n=0}^{\infty} p(5n+4)q^n \equiv \frac{1}{(q)_{\infty}^5} + \frac{1}{(q^5)_{\infty}}.$$

Thus, now extract every power  $q^{5n}$  from this identity, and equate the resulting series modulo 2. We obtain

$$q \sum_{n=0}^{\infty} p(25n+24)q^{5n+4} \equiv \sum_{n=0}^{\infty} p(n)q^{5n} + \sum_{n=0}^{\infty} p_5(5n)q^{5n}.$$

Making the substitution  $q^5 \rightarrow q$ , we then easily get:

$$q \sum_{n=0}^{\infty} p(25n+24)q^n \equiv \sum_{n=0}^{\infty} p(n)q^n + \sum_{n=0}^{\infty} p_5(5n)q^n.$$

This is equivalent to (15), thus completing the proof of case  $(5, 0, 5)$ .

Case  $(3, 8, 3)$  of Theorem 4 can be proved by 3-dissecting (11). Indeed, if we apply (11) to itself and then expand the  $q^{3n+2}$  terms of the denominator, we get

$$\sum_{n=0}^{\infty} p_3(9n+8) \equiv q^2 \frac{(q^3)_{\infty}^{36}}{(q)_{\infty}^{39}}$$

(see also [36], Theorem 1.2). Then, by a repeated application of (12) (the first time by substituting  $q \rightarrow q^4$ ), standard algebraic manipulations yield

$$\begin{aligned} q^2 \sum_{n=0}^{\infty} p_3(9n+8)q^n &\equiv q^4 \frac{(q^3)_{\infty}^{36}}{(q)_{\infty}^{39}} \equiv (q^3)_{\infty}^{36} (q)_{\infty}^9 \left( \frac{q^4}{(q^4)_{\infty}^{12}} \right) \\ &\equiv (q^3)_{\infty}^{36} (q)_{\infty}^9 \left( \frac{1}{(q^{12})_{\infty}^{12}} + \frac{1}{(q^4)_{\infty}^9 (q^{12})_{\infty}^9} \right) \\ &\equiv \frac{1}{(q)_{\infty}^{27}} + \frac{(q)_{\infty}^9}{(q^3)_{\infty}^{12}} \equiv \frac{1}{(q)_{\infty}^{27}} + \frac{1}{(q^3)_{\infty}^9} + \frac{q}{(q)_{\infty}^3}, \end{aligned}$$

which is the desired congruence for  $(3, 8, 3)$ .

Finally, case  $(3, 0, 9)$  is proved similarly to case  $(5, 0, 5)$ , by combining our result for  $(3, 8, 3)$  above with that for  $(3, 2, 3)$ , so we will omit the details.  $\square$

### 3. MODULAR FORM PROOFS

In this final section, we prove the cases  $(11, 6, 1)$ ,  $(17, 5, 1)$ ,  $(19, 4, 1)$ ,  $(23, 1, 1)$ , and  $(7, 1, 3)$  of Theorem 3. All of these — and, in fact, all cases proved in this paper — can be shown in a unified fashion using the machinery introduced by Radu in [28] to construct and verify a new class of modular form identities of the Ramanujan-Kolberg type.

Our general strategy is as follows (please see below for the relevant definitions): We first find an  $\eta$ -quotient which, when multiplied by  $\sum_{n=0}^{\infty} p(an+b)$ , yields a modular function, i.e., a modular form of weight zero. (Whether such an  $\eta$ -quotient exists is a separate question; should there exist a vector which satisfies certain conditions, Radu's theorem implies that this  $\eta$ -quotient does exist and has exponents given by the entries of the vector.) We then apply a classical theorem of Sturm (see [33]) to verify the conjectured congruence.

We first require a number of standard facts on modular forms, which we briefly recap here without proof for the reader's convenience. For background and proofs, we refer for instance to [16, 23].

We consider modular forms  $f$  of weight  $k$  and character  $\chi$  for  $\Gamma_0(N)$ , where this latter is defined as the subgroup of  $\text{SL}(2, \mathbb{Z})$  of those matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  such that  $c \equiv 0 \pmod{N}$ . The *level* of  $f$  is then the least value of  $N$  for which  $f$  is a modular form of weight  $k$  for  $\Gamma_0(N)$ . Note that in such a case,  $f$  is also a modular form of weight  $k$  for any  $\Gamma_0(cN)$ , where  $c \in \mathbb{N}^+$ .

An  $\eta$ -quotient is a quotient of functions of the form

$$\eta(z) = q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1 - q^i),$$

where  $q = e^{2\pi iz}$ . We recall that an  $\eta$ -quotient is a modular form under the following conditions, given by a theorem of Gordon-Hughes [12] and Newman [20].

**Theorem 9** ([12, 20]). *Let  $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ , with  $r_\delta \in \mathbb{Z}$ . If*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

*then  $f(z)$  is a modular form of weight  $k = \frac{1}{2} \sum r_\delta$ , level  $N$ , and character  $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$ , where  $s = \prod_{\delta|N} \delta^{r_\delta}$  and  $\left(\frac{\cdot}{d}\right)$  denotes the Jacobi symbol.*

Given two modular forms of weight  $k$  for  $\Gamma_0(N)$ , the following crucial result of Sturm [33] gives a criterion for determining when all of their coefficients are congruent modulo a given prime.

**Theorem 10** ([33]). *Let  $p$  be a prime number, and  $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$  and  $g(z) = \sum_{n=n_1}^{\infty} b(n)q^n$  be modular forms of weight  $k$  for  $\Gamma_0(N)$ , where  $n_0, n_1 \in \mathbb{Z}$ . If*

$$a(n) \equiv b(n) \pmod{p} \quad \text{for all } n \leq \frac{kN}{12} \cdot \prod_{d \text{ prime}; d|N} \left(1 + \frac{1}{d}\right),$$

*then  $f(z) \equiv g(z) \pmod{p}$  (i.e.,  $a(n) \equiv b(n) \pmod{p}$  for all  $n \in \mathbb{Z}$ ).*

It happens that all of the functions of interest to us in this paper yield modular forms of weight zero and so, regardless of  $N$ , Sturm's theorem is satisfied if all coefficients of the nonpositive powers of  $q$  are congruent.

Our primary tool is to suitably apply Radu [28], Theorem 45. We restate it here to keep the paper self-contained, which will require the following notation and definitions.

Let  $R(N)$  denote the set of integer sequences with entries  $r_\delta$  indexed by the positive divisors of  $N$ . For  $r = (r_\delta)_{\delta|N} \in R(N)$ , define

$$w(r) = \sum_{\delta|N} r_\delta; \quad \sigma_\infty(r) = \sum_{\delta|N} \delta r_\delta; \quad \sigma_0(r) = \sum_{\delta|N} \frac{N}{\delta} r_\delta; \quad \Pi(r) = \prod_{\delta|N} \delta^{|r_\delta|}.$$

Now denote by  $\Delta^*$  the set of tuples  $(m, M, N, t, (r_\delta)) \in (\mathbb{N}^+)^3 \times \mathbb{N} \times R(M)$  such that the following conditions are simultaneously satisfied: For all primes  $p$ , if  $p|m$  then  $p|N$ ; if  $\delta|M$  and  $r_\delta \neq 0$ , then  $\delta|mN$ ;  $t \in \{0, \dots, m-1\}$ ; finally, if  $\kappa = \gcd(1 - m^2, 24) \geq 1$ , then

$$24 \mid \frac{\kappa m N^2 \sigma_0(r)}{M}; \quad 8 \mid \kappa N w(r); \quad \frac{24m}{\gcd(\kappa(-24t - \sigma_\infty(r)), 24m)} \mid N;$$

and if  $m$  is even and  $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$  with  $j$  odd, then either  $4|\kappa N$  and  $8|Ns$ , or  $s$  is even and  $8|N(1-j)$ . (This latter condition is included for completeness but never applies in this paper.)

Let  $g_{m,t}(q)$  be the  $(mn+t)$ -dissection of the  $\eta$ -quotient defined by the exponents  $r_\delta$ : That is, for a given  $r = (r_\delta) \in R(M)$ , let  $\sum_{n=0}^{\infty} a_r(n)q^n = \prod_{\delta|M} \prod_{i=1}^{\infty} (1 - q^{\delta i})^{r_\delta}$ , and define

$$g_{m,t}(q) = q^{\frac{24t + \sigma_\infty(r)}{24m}} \sum_{n=0}^{\infty} a_r(mn+t)q^n.$$

For  $m, M \in \mathbb{N}^+$ , now let

$$P_{m,r}(t) = \left\{ \left[ ta^2 + \frac{a^2 - 1}{24} \sigma_\infty(r) \right]_m : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M) \right\},$$

where  $[n]_m$  denotes the least nonnegative integer congruent to  $n \pmod{m}$ . Finally, set

$$\chi_{m,r}(t) = \prod_{u \in P_{m,r}(t)} \exp((1 - m^2)(24u + \sigma_\infty(r))/(24m)),$$

where as usual  $\exp(x) = e^{2\pi i x}$ . Given these definitions, Radu's theorem is as follows.

**Theorem 11** ([28]). *Let  $(m, M, N, t, r = (r_\delta)) \in \Delta^*$ ,  $s = (s_\delta) \in R(N)$ , and  $\nu$  an integer such that  $\chi_{m,r}(t) = \exp(\nu/24)$ . Then*

$$F(s, r, m, t)(z) = \prod_{\delta|N} \eta^{s_\delta}(\delta z) \prod_{u \in P_{m,r}(t)} g_{m,u}(q)$$

*is a modular form of weight zero for  $\Gamma_0(N)$  if and only if the following conditions hold:*

$$(16) \quad |P_{m,r}(t)| \cdot w(r) + w(s) = 0;$$

$$(17) \quad \nu + |P_{m,r}(t)| \cdot m\sigma_\infty(r) + \sigma_\infty(s) \equiv 0 \pmod{24};$$

$$(18) \quad |P_{m,r}(t)| \cdot \frac{mN\sigma_0(r)}{M} + \sigma_0(s) \equiv 0 \pmod{24};$$

$$(19) \quad \left( \prod_{\delta|M} (m\delta)^{|r_\delta|} \right)^{|P_{m,r}(t)|} \cdot \Pi(s) \text{ is a square.}$$

We are now ready to prove the remaining congruences left open in the previous section.

*Proof.* We show in detail the cases (11, 6, 1) and (7, 1, 3) of Theorem 3. The other congruences are then simply a matter of finding the appropriate  $\eta$ -quotients, which we will list at the end for the reader.

We begin with case (11, 6, 1). We have  $m = 11$ ,  $M = 1$ ,  $N = 22$ ,  $t = 6$ , and  $(r_1) = (-1)$ . We verify that this  $(m, M, N, t, r)$ -vector exists in  $\Delta^*$ .

Consider now  $ta^2 + \frac{a^2-1}{24}\sigma_\infty(r) \pmod{m}$ , i.e.,  $6a^2 - \frac{a^2-1}{24} \pmod{11}$ . Since  $24^{-1} = 6$  in the field  $\mathbb{F}_{11}$ , we have that this expression is identically 6 for all  $a$ . Thus,  $P_{m,r}(t) = \{6\}$ .

Now we move to the conditions in Theorem 11. We notice that it suffices to choose  $\nu = 24$ , and that we need a vector  $s$  such that  $\omega(s) = 1$ ,  $\sigma_\infty \equiv 11 \pmod{24}$ ,  $\sigma_o \equiv 2 \pmod{24}$ , and such that  $11 \cdot \Pi(s)$  is a square. One can compute that  $s = (s_1, s_2, s_{11}, s_{22}) = (10, 2, 11, -22)$  satisfies the above conditions (here the divisors indexing the entries are the divisors of 22 in increasing order: 1, 2, 11, 22).

Therefore, we can construct

$$F(s, r, 11, 6) = q^{\frac{13}{24}} \frac{\eta(z)^{10} \eta(2z)^2 \eta(11z)^{11}}{\eta(22z)^{22}} \sum_{n=0}^{\infty} p(11n+6) q^n.$$

Recall that the congruence that we want to show is

$$q \sum_{n=0}^{\infty} p(11n+6) q^n \equiv \frac{1}{(q)_{\infty}^{11}} + \frac{1}{(q^{11})_{\infty}},$$

which, in terms of  $\eta$ -products, becomes

$$(20) \quad q \sum_{n=0}^{\infty} p(11n+6) q^n \equiv \frac{q^{\frac{11}{24}}}{\eta(z)^{11}} + \frac{q^{\frac{11}{24}}}{\eta(11z)}.$$

Multiplying through (20) by the  $\eta$ -quotient required to construct  $F$ , on the left-hand-side we obtain

$$q^{\frac{13}{24}} \frac{\eta(z)^{10} \eta(2z)^2 \eta(11z)^{11}}{\eta(22z)^{22}} \sum_{n=0}^{\infty} p(11n+6) q^n.$$

By Theorem 9, this is a modular form of weight zero. Similarly, the right-hand-side becomes

$$\begin{aligned} \frac{\eta(z)^{10} \eta(2z)^2 \eta(11z)^{11}}{\eta(22z)^{22}} \left( \frac{1}{\eta(z)^{11}} + \frac{1}{\eta(11z)} \right) &\equiv \frac{\eta(2z)^2 \eta(11z)^{11}}{\eta(z) \eta(22z)^{22}} + \frac{\eta(z)^{10} \eta(2z)^2 \eta(11z)^{10}}{\eta(22z)^{22}} \\ &\equiv \frac{\eta(4z) \eta(11z)^{11}}{\eta(z) \eta(44z)^{11}} + \frac{\eta(z)^{10} \eta(2z)^2 \eta(11z)^{10}}{\eta(22z)^{22}}, \end{aligned}$$

where in the last step we have used the fact that, modulo 2,  $\eta(z)^2 \equiv \eta(2z)$ .

By Theorem 9, both of these terms are weight zero modular forms. The sum of two such modular forms is also a modular form of weight zero (possibly for  $\Gamma_0(N)$  with  $N$  the least common multiple of the  $N$  for each form separately), so we can apply the Sturm bound of Theorem 10. In this particular case, since the weight of the functions is zero, we need only verify the coefficients of nonpositive degrees. One can quickly check, e.g. in Mathematica, that such coefficients on both sides are indeed congruent modulo 2. Therefore, the bound is satisfied and (20) is proved.

The idea to show the case  $(7, 1, 3)$  is similar and we will only sketch the argument here. Notice that now it suffices to choose  $m = 7$ ,  $M = 1$ ,  $N = 14$ ,  $t = 1$ , and  $(r_1) = (-3)$ . Again, selecting  $\nu = 24$  meets the requirements set forth, and we easily have that  $P_{m,r}(t) = \{1\}$ .

Picking  $s = (10, 10, 5, -22)$ , we see that we want to multiply the desired congruence for  $(7, 1, 3)$  by

$$q^{-\frac{21}{24}} \frac{\eta(z)^{10} \eta(2z)^{10} \eta(7z)^5}{\eta(14z)^{22}}.$$

This yields

$$q^{\frac{3}{24}} \frac{\eta(z)^{10} \eta(2z)^{10} \eta(7z)^5}{\eta(14z)^{22}} \sum_{n=0}^{\infty} p_3(7n+1) q^n \equiv \frac{\eta(4z)^3 \eta(7z)^5}{\eta(z) \eta(2z) \eta(28z) \eta(56z)^5} + \frac{\eta(z)^{10} \eta(2z)^{10} \eta(7z)^2}{\eta(14z)^{22}}.$$

A computation in Mathematica now quickly confirms that the right- and the left-hand-side are congruent modulo 2 in all nonpositive degrees, and thus, by Sturm's Theorem 10, the proof of  $(7, 1, 3)$  is complete.

For the remaining cases of Theorem 3, namely  $(17, 5, 1)$ ,  $(19, 4, 1)$ , and  $(23, 1, 1)$ , one can use (among other possible choices) the  $s$ -vectors  $(16, 2, 17, -34)$ ,  $(18, 2, 19, -38)$ , and  $(22, 2, 23, -46)$ , respectively.  $\square$

We mentioned earlier that all the cases of Theorems 3 and 4 could in fact be proven by this method. We provide below a list of corresponding  $s$ -vectors; for the cases of Theorem 3, we have  $N = 2m$ , so the divisors for the  $s$ -vector are  $1, 2, m, 2m$ . For the two cases of Theorem 4, we have  $N = 27$  and  $N = 125$ , and therefore the divisors are  $1, m, m^2, m^3$ . Given these facts, it is then a matter of verifying that all conditions are satisfied, making any necessary modulo 2 reductions to suit the conditions of Theorem 9, and then, using Sturm's bound, showing the resulting congruences by checking the coefficients of nonpositive degrees.

$(a, b, t)$	$s$ -vector
Any $(m, b, 1)$ in Theorem 3	$(m - 1, 2, m, -2m)$
$(3, 2, 3)$	$(6, 6, 9, -18)$
$(5, 2, 3)$	$(10, 8, 1, -16)$
$(7, 1, 3)$	$(10, 10, 5, -22)$
$(5, 0, 5)$	$(5, 1, 4, -5)$
$(3, 0, 9)$	$(9, 3, 6, -9)$
$(3, 8, 3)$ in Theorem 4	$(3, 1, 8, -9)$
$(5, 24, 1)$ in Theorem 4	$(5, 4, 2, -10)$

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